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THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer.

Papers intended for a particular Department should be sent to the Chairman of that Department.

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To Our Subscribers and the General Mathematical Public

With this the April, 1944, issue of National Mathematics Magazine two volumes of it have been nearly completed since, on July 1, 1944, the undersigned and his editorial colleagues assumed responsibility for its continued publication. During this period the undersigned, by allowing his name to be carried on the title page as its official publisher, has become automatically the party responsible for all debts incurred in its administration. This has meant, and could only mean, that, with the aid of our loyal colleagues, we have assumed the initiative in planning ways and means by which the Magazine revenues, through subscriptions, advertisements and donations, could be increased.

We have not the slightest intention of painting on this editorial page a picture of the obstacles, handicaps and discouragements that have been encountered during these two years. Even if such a picture were relevant to the purpose of this announcement, the painting of it should more properly be done by some one else.

During the current volume-year no large donation to the cause of the Magazine (such as the $400.00 voluntary donation by the Mathematical Association of America in the latter months of 1942) has been made. But a number of smaller donations have been generously tendered by individual friends of the cause—friends whose names compose a Roll of Honor which should be inscribed on the pages of this journal.

A conservative count of our roster of subscribers discloses that there are presently at least 50% more paid-up subscriptions than there were in the spring of 1942—a few months prior to the discontinuation of Louisiana State University sponsorship.

In order that the printing debt be held to the lowest possible figure during a period in which the anticipated revenues must, largely, come from its subscriptions, advertisements and the pocket of the publisher, it seemed advisable (1) to cut down the content of the journal from 48 or 52 pages to 40 or 44 pages, (2), to reduce the per-page cost of printing by temporarily withholding from publication many articles which had been approved by our various Committees, but which would be very expensive to set up in type. By adopting and carrying out these two policies, at least in an approximately thorough manner, we are able to predict, with reasonable confidence, that, after the publication of the May issue of Volume XVIII, our printing debt will be little if any more than it was at the same period last year.

The debt of last year was quite completely annihilated by the end of October, 1943, though it must be admitted that this was done very largely by applying to the debt moneys received for new and renewal subscriptions covering later periods.

In view of the above considerations and of others which could be cited, among which are the materially increased costs of all factors entering into war-time publishing enterprises, and, in order that NATIONAL MATHEMATICS MAGAZINE may be released from embarrassing handicaps; in order that the Administration of this journal shall be in definite position to publish as rapidly as possible the articles which the fine activity of the Chairmen of our different Committees has committed to our files—a gratifying number—; and, finally, in order that we shall be able to return to the issuing of a standard content for each number of the MAGAZINE—a content which shall be not less than 56 pages, we make the following

ANNOUNCEMENT

I. For all subscription periods beginning with the October, 1944 or a later issue of NATIONAL MATHEMATICS MAGAZINE, the subscription price to all subscribers shall be $3.00, if the remittance for said subscription is received at the Baton Rouge office after October 1, 1944.

II. If the remittance from a NEW subscriber to cover a subscription period beginning with or after the October issue of Volume XIX (whether said period covers one, or two, or more years) is received at the Baton Rouge office PRIOR to October 1, 1944, the amount remitted may be based on the OLD subscription rate of $2.00 per year.

III. All paid-up subscribers will be allowed the privilege of renewing their subscriptions on the basis of $2.00, provided their remittance checks are received at the Baton Rouge office before October 1, 1944.

S. T. SANDERS.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

An Eighth Lesson in the History of Mathematics

By G. A. MILLER
University of Illinois

17. *Real Numbers.* The concept of abstract real integers and fractions appears in some of the oldest mathematical records. For instance, in the Egyptian Rhind Mathematical Papyrus we find already the addition of the numbers of objects which differ widely, such as houses, cats, mice, etc. We also find here special symbols for abstract fractions, such as $^1/_2$, $^2/_3$, etc., as was noted in the first of these Lessons, Volume 13, page 272, (1939). The real numbers had two principal sources, viz., counting and measuring. In the former operations the units are often different, but these differences are overlooked in this operation and are often under the control of man. In measuring, on the contrary, man usually controls the units and adopts different units under different conditions. It is in this connection that the use of common fractions appeared very early, and the names of units of measures were sometimes the same as those of functions.

Since both the integers and the common fractions are of prehistoric origin, it is not a question of the modern history of mathematics to decide which are the older. although it has often been assumed that the integers are the older. Both common fractions and integers were used in special cases long before the appearance of a general number system. It is commonly assumed that the widespread use of the decimal system of notation and numeration is due to the fact that man early counted by means of his fingers, but, as this is also of prehistoric origin, it lies outside the domain of actual mathematical history, although the collections of evidences in the support of this theory belong to mathematical history, and these evidences are numerous. A considerable number of other bases of number systems have been in use, including the base 20, which appears extensively in the

number words of different languages and may have resulted from counting on the fingers and toes.

Since the extension of the number concept so as to include negative numbers met with considerable opposition even in recent times,* it may be desirable to emphasize here one of the difficulties encountered in the introduction of the modern concept of negative numbers. In the consideration of positive numbers, only the ratio of a larger number to a smaller number is always greater than unity and the ratio of a smaller number to a larger number is always less than unity. On the contrary, when we use negative numbers in the modern way we have such equations as $^2/_{-3} = -^2/_3$. Hence when negative numbers are used the ratio of a larger number to a smaller number may be the same as the ratio of a smaller number to a larger number. Since the ancient Greeks in their mathematical work emphasized the consideration of ratios, it is clear that the change of view along this liine, which was necessary in order to adopt the negative numbers as actual numbers, may have been an important obstacle in the way of this adoption.

While in the consideration of two unequal positive numbers the ratio of the larger to the smaller is always greater than unity, in the consideration of two unequal negative numbers the ratio of the larger to the smaller is always less than unity. The long struggle of the negative numbers to receive complete recognition as numbers cannot be fully understood unless we bear in mind that this recognition involved disadvantages as well as advantages. The latter furnished the motives which tended towards this recognition and appear to us now as so important that it may seem almost incredible that it was so long delayed. It met with strong opposition in the eighteenth century and even in the beginning of the nineteenth century, as results, for instance, from the writing of the noted Frenchman, L. N. M. Carnot, Cf. Encyclopedie des Science Mathematiques, tome 3, volume 2, (1813). It is stated here that the real reason which inspired the researches of L. N. M. Carnot was his aversion to negative numbers.

A fundamental fact in regard to negative numbers is that their occasional correct use is much older than their correct theory. The ancient Babylonians used the terms *tab* and *lal* in the same way that we now use + and − to represent distances in opposite directions from a fixed straight line, and, in medieval times it was recognized that if a positive number represents a credit the corresponding negative number may represent a debt. Hence it may appear somewhat striking that

*In the preface F. F. Goddy's *Interpretation of the Atom* (1932), the author said "I have tried, in other fields, to show the incredible confusions, of which the whole world is now one seething example, that have followed from the invention by the Hindu mathematicians of negative quantities, and their justification from their anology to debts."

the first known instance of the solution of a quadratic equation with two negative roots appeared in 1629, as was noted in the third of these Lessons, published in this Magazine, volume 15, pages 234-244 (1941). It should be emphasized that the history of the use of negative numbers as actual numbers can be largely determined from the use of these in the early solutions of quadratic equations. In particular, no instances have yet been found of the use of a negative root by the Ancient Greeks in their partial solutions of quadratic equations. They did not really master this equation.*

While the use of negative numbers as roots of an equation is more recent than might have been expected, it should be explicitly noted that such numbers were used in equations long before they were used as roots. The ancient Babylonians gave equations in which one member of the equation is a negative number, as may be seen by consulting O. Neugebauer's Mathematische Keilschrift-texte, page 38 (1935). The difference between the use of negative numbers in addition and subtraction and the use of these numbers in multiplication and division should be emphasized. The ancient Babylonians already observed that negative numbers may be regarded as subtracted numbers. Since subtraction is the inverse of addition we have merely an operation and its inverse is the use of positive and negative numbers in addition and subtraction. On the other hand, the inverse of multiplication is division. Hence the inverse operation here gives rise to the reciprocals of numbers but not to negative numbers.

The rule that the product of two numbers is a positive number was noted already by the ancient Babylonian Astronomers at least as early as the fourth century B. C. Diophantus later stated it approximately only when he asserted that the product of two subtracted numbers is an added number since he implied that the minuend exceeds the subtrahend in these cases and hence he did not deal with numbers which were actually negative. Several centuries later the Hindus dealt with actually negative numbers and observed that a positive number has two square roots which differ only in sign. It is possible that here as in many other cases greater generality at first was achieved by a lack of keen insight rather than by deeper insight. Mathematical advance based on generalizations has probably in many cases been due to stupidity, but this fact can usually not be definitely established because words frequently do not express the ideas which were in the writer's mind.

Various motives which tended towards the general introduction of negative numbers were considered by the present writer in an article which appeared in L'Enseignement Mathematique, Volume 33,

*For an opposite statement, see D. E. Smith, *History of Mathematics*, p. 126.

pages 221-226 (1934). Among these is the fact that if we consider the various powers of a positive integer which exceeds unity then the negative exponents correspond to the positive numbers which are less than unity while the positive exponents correspond to the numbers which exceed unity. When logarithms were introduced it therefore became very desirable to use negative exponents as well as zero and positive exponents. Another powerful motive was furnished by the system of rectilinear coordinates when analytic geometry came into general use. R. Descartes used negative ordinates but not negative abscissas, 1637. The study of the trigonometric functions and the solution of triangles were also greatly simplified by the use of negative numbers. It should, however, be emphasized that the introduction of negative numbers is a convenience rather than a logical necessity. This difference is of considerable mathematical interest.

In many cases positive integers were supplemented very early by numbers congruent with respect to certain moduli, as when the days were first expressed in terms of the number of days in a week, month, or year. The theory of congruences is therefore older than the introduction by C. F. Gauss (1777-1855) of the symbol $\equiv$ to express congruences. In fact congruences constitute some of the oldest steps towards an intellectual penetration into our surroundings. The hours of the day represent another effort to simplify our insight by means of the concept of congruences. This is also illustrated by the use of various units of measure such that a certain number of the smaller units is equal to one larger unit. An emphasis on these obvious facts may throw light on the late general use of congruences in the theory of numbers and the emphasis which is frequently placed on the contributions of C. F. Gauss and others along this line. The difference between the uses of the term congruent in geometry and in number theory should be noted. Two plane triangles are said to be congruent when they are exactly equal to each other, while two positive number are said to congruent with respect to a modulus when they may differ widely otherwise.

An interesting fact in the history of real numbers is that the ancient Babylonians used the same symbol for different numbers which are integral powers of 60 including zero and its negative powers. All these powers constitute a group with respect to multiplication in the modern sense of the term group as used in mathematics. The Babylonians of course did not use negative exponents explicitly. This use appears in the Triparty by Chuquet (1484). The exponent zero also appears in this work with its modern meaning. The number *one* was not regarded as a number by many of the Greek writers, but merely as the source of numbers. Many of the later mathematicians

adopted this view as a result of the great respect for Greek authority. Stevin explicitly stated in his L'Arithmetique (1585) that *one is* a number, and this view was gruadually adopted by the later writers on the theory of numbers. The number zero naturally encountered still greater opposition than unity in its forward steps towards being accepted as an actual number.

A very important and lengthy chapter in the history of the extension of the real number concept is furnished by the development of the theory of irrational numbers. This history begins with the writings of the ancient Greeks, although they never generally accepted the view that there are such things as irrational numbers. They were, however, the first, as far as we know now, to recognize clearly that there are irrational geometric quantities, and that $\sqrt{2}$, for instance, is not a number which can be expressed in the form a/b where a and b are integers. It is not known when this fact was first discovered but it is known that before the times of Euclid the Greeks were greatly interested in it and that it led them to distinguish between numbers and the values of such geometric quantities as *lines**. There is obviously no good reason for assuming that numbers are discontinuous and that lines are continuous, but the ancient Greeks made this assumption and they greatly hindered the development of our subject for more than a thousand years by this assumed difference.

The Rhind Mathematical Papyrus, about 1700 B. C., is an important document for the history of positive real numbers. Fortunately it has been translated into English by T. E. Peet (1923) and A. B. Chace (1927). The latter translation was published by the Mathematical Association of America and hence is readily available to the American student of our subject. It begins with a table of abstract fractions in which the fractions equal to 2 divided by the various odd numbers from 5 to 101 are expressed as the sums of reciprocals of integers, commonly called unit fractions. For instance, $^2/_5 = ^1/_3 + ^1/_{25}$, $^2/_7 = ^1/_4 + ^1/_{28}$, etc. An extensive literature relating to it has appeared and it exhibits various wide differences from the methods of dealing with common fractions. It seems probable that the ancient Egyptians had a clear idea of a general common fraction notwithstanding their common use of unit fractions with the exception of $^2/_3$, and, in very early times, $^3/_4$ for which they had special symbols, as were noted above.

The use of unit fractions by the ancient Egyptians is especially interesting in view of the fact that it was followed later by many of the people who seem to have been influenced by its use in Egypt. A

*Tropfke, *Geschichte der Elementar Mathematik*, Volume 2, p. 933 (1833). It is stated here that Newton (1673) assumed the identity of number and line segment.

widely different method of dealing with fractions seems to have arisen in Asia, and was extensively used by the ancient Babylonians. This method was based on the use of 60 as a fundamental unit in dealing with numbers. Hence all of the special fractions $^1/_2$, $^1/_3$, $^1/_4$, $^1/_5$, $^1/_6$, could be expressed in the form of integers, somewhat like we now use decimal fractions. The Babylonians did, however, not use a symbol corresponding to our decimal point, and hence their notation for numbers was very indefinite from the modern point of view. It was noted above that they used the same symbol for the various powers of 60, which is an illustration of this indefinite notation and naturally excites our interest in their motives in this respect as well as their possible lack of insight.

The ancient Greeks were the first to discuss the number concept as such. Even the division of the positive integers into the two categories of odd and even appears for the first time explicitly in the Greek literature, although the ancient Egyptian table of 2 divided by some odd numbers, to which we referred above, seems to imply that they already were conscious of the possible classification of all positive integers in this way. It should be emphasized that some ancient Greeks really took a step backwards as regards the number concept, at least in their philosophical work, by confining this concept to positive integers notwithstanding the fact that common fractions had been used extensively by pre-Grecian mathematicians. Philosophers sometimes retarded the progress of mathematics. It was Diophantus (near the end of the third century A. D.) who seems to have first made definite use of rational numbers as numbers among the Greek mathematicians.

The Pythagoreans devoted considerable attention to the question of prime numbers and thus they directed attention to problems of number theory which have not yet been satisfactorily solved and have engaged the interest of prominent mathematicians since the times of the ancient Greeks. In Euclid's *Elements* it is proved that there is no upper limit to the number of the prime numbers since the production of all the prime numbers up to a certain given one, increased by unity, is either a prime number or is divisible by a larger prime number than any of those which had been given. This fact had probably been proved before the time of Euclid but it is not known who first proved it. It is probable that it is a Greek contribution to knowledge since there is no evidence that the pre-Grecian mathematicians were interested in questions which are only of theoretic interest. When the largest of these said prime numbers is 13 we thus obtain 30031, which is $59 . 509$, but when the largest given prime number is less than 13 there result only prime numbers by the use of this method.

The oldest known method to find all the prime numbers which are less than a given number is the so-called Sieve of Eratosthenes (276-194 B. C.) This method seems to have been known before the days of Eratosthenes but is probably of Greek origin. In a series of successive natural numbers beginning with unity one may first cancel every second, then every third, then every fifth, etc. The numbers which remain uncancelled are obviously the prime numbers in the series. While this method does not involve much mathematics it is the only one relating to this subject which has been transmitted to us from ancient times. To determine whether a given number n is a prime number it was customary to divide n by the different prime numbers which are less than $\sqrt{n}$, and no better method seems to have been known before the seventeenth century. Euclid gave already the fundamental theorem that if the product of two numbers is divisible by a prime number, then at least one of these factors is divisible by this prime. This generalization has been called the fundamental theorem of arithmetic and is one of the earliest very useful theorems in the theory of numbers.

The Bible does not contain any number which is as large as a million and in the third edition of Volume 1 of the well known *Geschichte der Mathematik*, by Moritz Cantor, it is stated (page 22) that the cuneiform writing of numbers by the Babylonians did not extend as far as a million; at least no such number had then been found. As similar statements have appeared in many other places, it is desirable to note here that it is now known that much larger numbers were used by the ancient Babylonians. The Greek writer Archimedes, who is commonly regarded as the greatest mathematician of antiquity, wrote a work called *The Sand-reckoner* in which he developed a system of numeration which is not only amply extnsive to provide in a brief form different numbers for all the visible grains of sand on earth, but which provides such a vast number of numbers that those required for this purpose are an insignificant part of the available ones. The multitude represented by "the sand of the sea" is therefore insignificant in comparison with the multitude of numbers described in this work of Archimedes, written more than two centuries before the beginning of the Christian era. Apollonius, who was also a very great Greek mathematician and lived about the same time devised a somewhat different system of numeration based on 10^4, while Archimedes used 10^8 as the base of his system. The Hindus and the Chinese also used very large powers of 10. It is said that the powers where as large as 10^{53}, but it should be noted that the dates relating to their early mathematical work are quite uncertain, since their claims have been found to be unreliable in many instances. There are however numerous

evidences of the fact that the ancients were greatly impressed by the rapid increase of powers of numbers with a large base. These are among the awe inspiring elements of the development of mathematics which were effective in arousing interest.

By raising positive integers to powers we associate two series which played a fundamental role in the development of mathematics since very early times, viz., the arithmetic and the geometric series. The great disparity between these series as regards the rapidity of increase naturally was observed very early. These series are at the base of logarithms, but this subject did not follow as directly from exponents as might be at first supposed to have been the case, and hence the history of logarithms is a much more difficult subject than the modern elementary theory of logarithms. The history of mathematics involves many other instances of difficulties encountered by early workers as a result of not finding at first the direct road leading to the main objects in view. This is very well illustrated by the efforts of J. Napier (1500-1617) and others to base logarithms on the general relation between arithmetic and geometric series instead of basing it directly on the theory of the exponents and powers of a number, now called the base of the system of logarithms.

If the six fundamental operations of arithmetic, already used in the 12th century by the Hindus, are arranged in the following order: addition, subtraction, multiplication, division, raising to powers, and extracting roots, then each of the last four is reduced by two steps by means of logarithms. This greatly contributed to the introduction of logarithms in the seventeenth century through the efforts of many writers, among whom J. Napier is usually given the greatest credit. The logarithms which he computed are however not those which are now in common use, and they are not the exponents of the powers of a given number. A very large number of different tables of logarithms have been published, but most of them have been copied from other tables which were assumed to be accurate. According to the *Encyclopedie des Sciences Mathematiques*, tome 1, Volume 4, page 288, more than 553 such distinct tables have been listed.

We referred above to the fact that there was at first a considerable difference of opinion in regard to the questions whether unity and zero should be regarded as actual numbers. It is obvious that these numbers have various exceptional properties and that their admission as actual numbers may evidently be debated even at the present time, and such debates may serve ro give a clearer insight into their exceptional properties. Division by zero is now very commonly excluded from the legitimate operations of algebra, but zero is now very commonly accepted as a root of certain equations. There are numerous

such debatable questions in mathematics which a study of the history of this subject naturally revives and clarifies by exhibiting which of two opposing views was later more widely used, even if no definite answer can be given to the question, which is the better view. In some cases this depends on the advancement of those using the subject.

An interesting question relating to real numbers is the fact that they were combined already in very early times according to the four operations of addition, subtraction, multiplication and division, while in theory of abstract groups, which came into extensive use during the nineteenth century, the elements are combined according to only one operation, which was at first generally called multiplication, but was later called addition. This directs attention to the fact that the theory of abstract groups is more general than the theory of numbers and it is a clear illustration of the difference between importance and generality of mathematical concepts. It is obvious that the number concept is more important than the group concept in the development of our subject, but that the group concept is the more general of the two. While the trend in the development of mathematics has been towards greater and greater generality it is desirable to bear in mind that generality and importance are not equivalent terms as regards the development of our subject. The solution of special cases has often led to general methods.

The pre-Grecian mathematics which has been transmitted to us is very weak in respect to general statements but it is commonly assumed that this work involves many general rules which were expressed in a special form. For instance, the Babylonians found the sum of the first 10 integers by a method which is in accord with the formula that the sum of the squares of the first n natural numbers is equal to the sum of these numbers multiplied by $(1+2n)/3$. Similarly, the use of arithmetic and geometric progression by the ancient Egyptians and the ancient Babylonians shows that they worked according to various general rules which are not explicitly formulated in any of their extant writings. The use of general rules is much older than the development of the mathematical notation which made it possible to use simple general formulas.

It may be desirable to direct attention here to an erroneous statement relating to positive integers, which appears on page 25 of *What is Mathematics?* by Courant and Robbins (1941) since this book has been widely used and hence its errors are of especial interest. It is stated here that "Fermat made the famous conjecture (but not the definite assertion) that all numbers of the form 2^{2^n} are primes". The truth is that Fermat (1601-1665) made this statement at first as a conjecture but later as a fact. In 1732 L. Euler (1707-1783) proved

that $2^{32}+1$ is composite. Fermat is perhaps best known in the history of mathematics on account of the fact that he announced the theorem that $a^n+b^n=c^n$ cannot be satisfied by positive integral values of a,b,c when $n>2$. A prize of 100,000 marks was offered in Germany to the person who would first give a solution and it was held in trust by the Royal Academy of Göttingen. Until the post-war inflation wiped out the monetary value of this prize a very large number of incorrect supposed solutions were published in various countries. Cf. Miller's *Historical Introduction to Mathematical Literature*, page 153 (1916).

There is probably no other mathematical subject in which the consideration of published errors is more instructive than in the history of mathematics since this history involves very many questiohs which require minute distinctions in order to arrive at actual situations. To evaluate properly the actual contributions made by the various writers on our subject requires not only a deep insight into the subject but also the ability to evaluate the earlier contributions. Such common brief statements as that Descartes discovered analytic geometry and Newton discovered the calculus may have some value, but they reveal little as regards the ignorance of these great mathematicians in respect to many phases of the wonderful subjects of the modern analytic geometry and the modern calculus, which were developed by thousands of different writers. This ignorance is also instructive because otherwise it is not possible to obtain a true picture of the mathematical knowledge on the part of the human race during recent centuries. The real numbers have a long and inspiring history in the gradual enlightenment of the human race both as independent elements and as elements entering into progress of a great variety of types.

The wide usefulness of real numbers in business affairs is partly exhibited by the extensive use of calculating machines. This use is comparatively modern if we exclude the use of the simple abacus. In the seventeenth century the noted French mathematician B. Pascal (1623-1662) constructed such a machine which is said to have attracted much attention in Paris, and somewhat later the noted German mathematician G. W. Leibniz (1646-1716) spent a large amount of his own money in his efforts to perfect such a machine. His lack of complete success has been attributed to insufficient mechanical skill in his time. The use of numerical machines in modern times is not restricted to business computations. They have been used for many other purposes.*

*Cf. *Punched Card Methods in Scientific Computation*, by W. J. Eckert (1940).

The Teachers' Department

Edited by
WM. L. SCHAAF, JOSEPH SEIDLIN, L. J. ADAMS, C. N. SHUSTER

On the Summation of Certain Types of Finite Series

By A. M. NIESSEN
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The calculus of finite differences offers convenient methods of finding the sum of certain finite series. A. B. Farnell* showed the application of the theorem

(I) $$\sum_{x=1}^{n} \Delta F(x) = F(n+1) - F(n)$$

to the summation of the series

$$(a_0 + a_1 x + \cdots + a_k x^k).$$

This note will discuss other application of the methods of finite differences to the summation of several interesting types of series.

We shall use the following notations:

(a) u_k will denote the kth term of the series

(b) $\Delta f(x) = f(x+1) - f(x)$

(c) $k^{(r)} = k(k-1)(k-2)\cdots(k-r+1)$

(d) $k^{(-r)} = \dfrac{1}{k(k+1)\cdots(k+r-1)}$

(e) $P(x) = a_0 + a_1 x + \cdots + a_r x^r$

(f) $[f(x)]_1^{n+1} = f(n+1) - f(1)$

Theorem 1. Let $f(x) = \Delta F(x)$, then

(1) $$\sum_{x=1}^{n} f(x) = [F(x)]_1^{n+1}$$

*Summation of Finite Series with Polynomial Terms, NATIONAL MATHEMATICS MAGAZINE, November, 1942.

For proof see A. B. Farnell, *loc cit.*

Theorem 2.

$$\sum_{k=1}^{n} k^{(r)} = \frac{1}{r+1}\left[k^{(r+1)}\right]_1^{n+1}. \tag{2}$$

Proof: By definitions (b) and (c) we obtain

$$\Delta k^{(r+1)} = (r+1)k^{(r)},$$ hence

$$k^{(r)} = \frac{1}{r+1}\ \Delta k^{(r+1)}$$

Applying Theorem 1 we have

$$\sum_{k+1}^{n} k^{(r)} = \frac{1}{r+1}\ \sum_{k=1}^{n} \Delta k^{(r+1)} = \frac{1}{r+1}\left[k^{(r+1)}\right]_1^{n+1}.$$

Example: $S_n = 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n(n+1)(n+2)$

Here $u_k = (k+2)^{(3)}$, consequently

$$S_n = \tfrac{1}{4}\left[(k+2)^{(4)}\right]_1^{n+1} \tag{2,1}$$

Theorem 3.

$$\sum_{x=1}^{n} P(x) = nP(0) + \left[\frac{\Delta P(0)}{2!}\ x^{(2)} + \cdots + \frac{\Delta^r P(0)}{(r+1)!}\ x^{(r+1)}\right]_1^{n+1} \tag{3}$$

Proof: If we put in Newton's advancing differences formula:

$$f(x+nh) = f(x) + \frac{n^{(1)}}{1}\ f(x) + \frac{n^{(2)}}{2}\ f(x) + \cdots$$

$x=0$, $n=x$, and $h=1$, we obtain

$$P(0+x) = P(0) + \frac{x^{(1)}}{1!}\ \Delta P(0) + \cdots + \frac{x^{(r)}}{r!}\ \Delta^r P(0)$$

for in the case of a polynomial of degree r the rth differences will be constant and all $(r+i)$th differences will be zero for any index $i = 1, 2 \cdots$.

Applying Theorem 2 to each term of the above expansion we obtain the result of Theorem 3.

Example:

$$S_n = (3\cdot 1^2 - 4\cdot 2 + 2) + (3\cdot 2^2 - 4\cdot 2 + 2) + \cdots + (3n^2 - 4n + 2)$$

$$P(k) = 3k^2 - 4k + 2$$

k	$P(k)$	$P(k)$	$P(k)$
0	2	-1	6
1	1	5	6
2	6	11	
3	17		

Here $\quad P(0)=2, \quad P(0)=-1, \quad P(0)=6$

$$S_n=2n+\left[\frac{-\,k(k-1)}{2}+\frac{6k(k-1)(k-2)}{6}\right]_1^{n+1} \tag{3,1}$$

$$=\frac{n(n+1)(2n-3)}{2}+2n.$$

Theorem 4. For all $r=2,3\cdots$

$$\sum_{k=1}^{n} k^{(-r)}=\frac{-1}{r-1}\left[k^{(-r+1)}\right]_1^{n+1} \tag{4}$$

Proof: By definitions (b) and (d) we obtain

$$\Delta k^{(-r+1)}=-(r-1)k^{(-r)}, \quad \text{for all } r=2,3\cdots\cdot$$

hence
$$k^{(-r)}=\frac{-1}{r-1}\,\Delta k^{(-r+1)}.$$

Applying Theorem 1 to the above result we obtain (4).

Example:

$$S_n=\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}.$$

Here $\quad u_k=k^{(-3)}=\dfrac{-1}{2}\,\Delta k^{(-2)}$

$$S_n=\frac{-1}{2}\left[\frac{1}{k(k+1)}\right]_1^{n+1} \tag{4,1}$$

$$=\tfrac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

Theorem 5.

$$\sum_{x=1}^{n} u(x)\Delta v(x) = [u(x)\ v(x)]_1^{n+1} - \sum_{x=1}^{n} v(x+1)\Delta u(x) \tag{5}$$

Proof: By definition (b)

$$\begin{aligned}\Delta[u(x)\ v(x)] &= u(x+1)\ v(x+1) - u(x)\ v(x)\\ &= v(x+1)[u(x+1)-u(x)] + u(x)[v(x+1)-v(x)]\\ &= v(x+1)\Delta u(x) + u(x)\Delta v(x)\end{aligned}$$

Hence $\quad u(x)\Delta v(x) = \Delta[u(x)\ v(x)] - v(x+1)\Delta u(x).$

Applying Theorem 1 to both members of the above equality we obtain (5).

Example:

$$S_n = e + 2e^2 + \cdots + ne^n$$

$$u_k = ke^k.$$

By definition (b) we obtain

$$\Delta e^k = (e-1)e^k, \qquad \text{or } e^k = \frac{1}{e-1}\,\Delta e^k.$$

Thus
$$u^k = \frac{1}{e-1}\,k\Delta e^k.$$

Applying (5) we obtain

$$\begin{aligned}S_n &= \frac{1}{e-1}[ke^k]_1^{n+1} - \frac{1}{e-1}\sum_{x=1}^{n} e^{k+1}\\ &= \frac{e}{e-1}[(n+1)e^n - 1] - \left(\frac{e}{e-1}\right)^2 (e^n - 1)\end{aligned} \tag{5,1}$$

The series k^2e^k and others of the same type can be handled in a similar manner.

We shall consider finally an example involving the summation of the harmonic series

$$H_n = 1/1 + 1/2 + 1/3 + \cdots + 1/n.$$

Let
$$S_n = \sum_{k=1}^{n} \frac{(2k-1)(2k+1)}{2k(2k+2)}.$$

The general term can be written in the form

$$u_k = \frac{(k-1/2)(k+1/2)}{k(k+1)} = -(k+1/2)\Delta\frac{1}{k}$$

$$S_n = -\left[k - \frac{1}{4k}\right]_1^{n+1} + 2\sum_{k=1}^{n} \frac{k+1/2}{k+1}$$

$$= -\left[k - \frac{1}{4k}\right]_1^{n+1} + \frac{n(2n+3)}{n+1} - \sum_{k=1}^{n} \frac{1}{k}.$$

The methods outlined above fail in the case of the harmonic series. The approximate value of the sum of the first n terms can be found by applying the theorem

$$\int_n^{n+p} f(x)dx < \sum_{j=0}^{p} f(n+j) < f(n) + \int_n^{n+p} f(x)dx \cdots *$$

Taking for instance $n=10$ and $p=90$, we obtain

$$\log_e 10 < \sum_{k=10}^{100} \frac{1}{k} < \log_e 10 + 0.1$$

for

$$\int_n^{n+p} \frac{dx}{x} = \log\left(1+\frac{p}{n}\right).$$

Thus
$$2.3026 < \sum_{k=10}^{100} \frac{1}{k} < 2.4026.$$

Taking $2.3026+0.05=2.3526$ as the value of the above sum and computing the sum of the first 9 terms

$$1/1+1/2+\cdots+1/9$$

directly we obtain the approximate value of the sum

$$\sum_{k=1}^{100} \frac{1}{k}.$$

*E. Goursat: *Course in Mathematical Analysis*, Vol. I, Par. 161.

Mathematics in the Open Forum

By Sister Helen Sullivan
Mt. St. Scholastica College

A global war, such as we are engaged in at the present time, has the general effect of making us look away from ourselves and our restricted spheres of activity into the larger spaces occupied by beings like ourselves and confronted with similar problems. That it does so is good. That it does so specifically in the field of mathematics is even better. This war has definitely awakened mathematicians to the fact that something has to be done now to insure, for posterity, mathematics in its pristine state of richness and many-sidedness. As we glance about us we find the same fear for the future of mathematics expressed in every quarter where mathematics is being studied and talked about.

"We Look Before and After,"* by Marion E. Stark of Wellesley College which recently appeared in these pages, found a very sympathetic response in the writer. Its message carried a note of challenge and at the same time some definite suggestions as to solution. Any mathematics professor who is alive is acutely aware that at the present time mathematics enjoys an unprecedented prominence. Only a casual glance at the publishers catalogs with their lists of books designed to serve as "refresher courses" in mathematics is needed to impress the reader with adequate evidence of the major role being played today by mathematics in the tragedy of a war-spent life. (We smile at the naivete displayed in the universal use of the word "refresher". Can it be that these firms are sincere in the belief that the basic facts of mathematics are common property and need only be refreshed in our immediate consciousness?)

The mathematician worthy of the name is also a philosopher, i. e., he traces immediate reality back to its first causes. Consequently he has gone deeper into the reasons for this popularity which prompts men today to qualify the present war with terms like "scientific" and "mathematical". Mathematics is popular in the sense that war is popular—it temporarily engages the attention of the masses becuase of necessity. The mathematician knows full well thta his beloved science parades in the lime-light simply and solely bedcause of its unquestionable utility. He shudders to think where this science will

*National Mathematics Magazine, Vol. XVIII (December, 1943) p. 116.

find itself when the war is won and the demand for immediate application in the realm of matter is removed.

The writer, who strongly upholds the idea and *practice* of liberal arts education because of its power to make man free to think, to choose and to love, has always taught mathematics with this in mind. If those who are privileged to remain in institutions devoted to the education of man in his entirety would fully imbibe the whole gospel that mathematics has to impart, our fears would be somewhat allayed. If we are not called upon to use our knowledge and skill directly, in the war effort, it is our unmistable patriotic duty to preserve and pass on the wealth of our mathematical heritage. This can only be done by an insistence on the recognition of powers peculiar to mathematics which *transcend* the level of computational skills.

The present age, more than any other, because of its unenduring, superficial culture, demands of us teachers that we inculcate the all-around character of mathematics. As loyal American citizens on the home front it is our duty to pass by no opportunity wherein the value of mathematics as a mode of thought, as the backbone of culture, as a practical computational science, can be demonstrated. We must capitalize on the fact that our classrooms are packed with students. While they stand holding out their hands for a minimum of meagre skills that will open the way for mechanical jobs (made attractive with high war-time wages attached) we must hasten to fill not only their hands but more especially their minds with the riches of mathematics in the realm of thought and endeavor.

The present paper describes the value of the open forum discussion as applied to mathematics. It has been a year's project with the mathematics group in this college and has met with sufficient success to warrant publication. It answers some of the questions raised by Miss Stark in the article previously mentioned. It is hoped that it will provoke comment and stimulate like action in other groups. It is described rather fully in what follows.

The topic chosen by the Kansas Gamma Chapter of Kappa Mu Epsilon (which name was given our mathematics club in 1940 when it passed from infancy to maturity) for the current school year was "Mathematics as it is Studied and Taught in Other Countries". For the November meeting it was decided to concentrate on the work of French-speaking peoples. Those members of the fraternity who have some knowledge of the language were delegated to conduct the first panel. Weeks of intense preparation were devoted to the project before it was presented to the group. French professors in this and other colleges were enlisted to help in the matter of locating suitable material. The response on the part of the language professors was

gratifying and implied, at least in part, this unspoken idea—"at last mathematics is going to get around and orientate itself and link up with other fields of knowledge". The librarians were whole-heartedly enthusiastic over the project and obtained books and other reference material from the State University and the library of Congress on the inter-library loan. The panel discussion was opened by the chairman with a clear concise statement of the project and a brief history of the French Academy and its more distinguished mathematical members. The first of the four speakers translated a portion of Descartes "La Geometrie" and gave explanations and demonstrations of the theorems translated. A second member handled the treatise on probability by Laplace entitled "Sur les Probabilites" in like manner. She illustrated her original translation with well chosen examples from everyday life. The life and work of Sophie Germain was the basis of the paper presented by the third member in the panel. The fourth member made a comparative study of old and new French mathematics texts and closed her talk with an account of the French school system with particular emphasis on the teaching of mathematics. Lively group discussion for a period of thirty minutes followed.

No one is ignorant of the amount of newspaper space and radio time that are being given to the subject of fostering a good-neighbor policy. With this thought uppermost in mind, the January meeting was set aside for a discussion of mathematics as it is treated in Spanish-speaking countries. For months in advance, letters had been written to various educational authorities and college professors in the South American countries. The chapter president had direct correspondence with Bernardo Baidaff, the editor of *Boletin Mathematico,* which is published in Buenos Aires. Guest speakers for the second meeting were two exchange-students from Mexico City who provided background material for the discussion by their talks on educational systems and tendencies in Mexico. As in the case of the first panel, students familiar with the Spanish language were put in charge of the discussions. One student gave direct translations from current issues of the *Boletin Mathematico* and gave us her results of a comparative study of the Spanish mathematics periodicals and those published in our own country. A second member of the panel translated Bernardo Baidaff's letter for the group and added other pertinent material concerning general mathematical policies in South America. Reports on the Mathematics Congress held in Mexico and on the work being carried on at the University of the Phillipines were given by a third member. The contributions of the fourth panel member consisted of a discussion of mathematics texts used in Spanish-speaking countries with clever illustrations pointing out singularities in ex-

pression of well-known mathematical facts. The third panel meeting was devoted to the work of British mathematicians, past and present. It was well done, and only scarcity of space prevents a full description. The fourth meeting is being planned for at this writing. The response on the part of the entire college has been most encouraging. The meetings were well attended by faculty members and students of diverse interests. The disccussions were lively and pointed. The writer feels that, in local circles at least, mathematics has definitely scored a point not for its utility but because it has shown its adaptability and its general relationship to other branches of learning.

The foregoing is an attempt to answer, at least in part, the stimulating challenge offered by Miss Stark. It is apparent that many other solutions could be advanced to supplement the project here described. How can we, at the present time, best "sell" our science so that when the emphasis brought on by war-time necessity gives way it will have taken so strong a hold on the minds and hearts of men that its very comprehensiveness and richness will have made it indispensable for complete living?

Brief Notes and Comments

Edited by
H. A. Simmons

Geometric Derivation of the Formula for Integration by Parts

Benham M. Ingersoll, Captain A. U. S.
United States Military Academy

Knowing that geometric visualizations are frequently of value in aiding students to comprehend and remember analytic results, I submit this brief remark in the hope that it will be instructive in the elementary *Calculus* course. Although it is hardly more than a casual observation, its inclusion here seems justified by the fact that a quick search through a half dozen widely used standard texts fails to reveal this simple device.

The formula for integration by parts as applied to the evaluation of the definite integral can be given as follows: If $u(x)$ and $v(x)$ are single-valued differentiable functions of x for x in the closed interval $[a, b]$, then

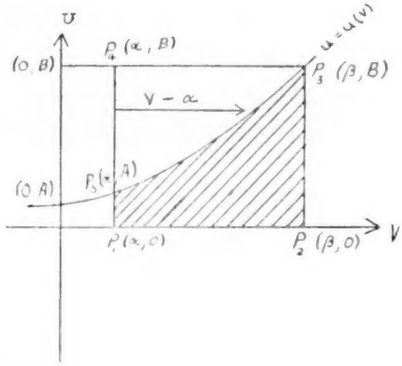
$$\int_{x=a}^{x=b} u\,dv = uv\Big]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du \tag{1}$$

To achieve maximum simplicity, we first suppose that both u and v are monotonically increasing non-negative functions of x in the interval $[a,b)$. Geometrically it is then obvious that the elimination of x from the equations $u=u(x)$, $v=v(x)$ yields u and v as a monotonically increasing non-negative functions of each other.

Writing now for $v(a)$, $v(b)$, $u(a)$, $u(b)$ the letters α, β, A, B, respectively, and drawing the curve $u=u(v)$, we have the following figure.

Clearly:

$$\text{Area } P_1P_2P_3P_5 = \text{Area } P_1P_2P_3P_4 - \text{Area } P_5P_3P_4.$$



Employing the areal interpretation of the definite integral, this yields:

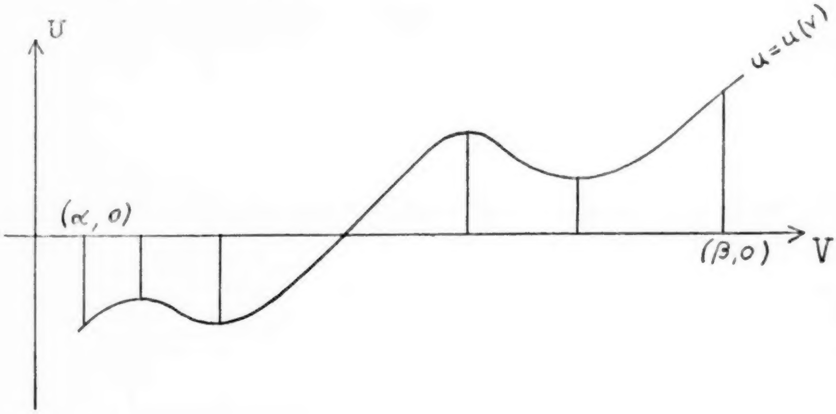
(2) $$\int_{\alpha}^{\beta} u\,dv = (\beta-\alpha)\cdot B - \int_{A}^{B} (v-\alpha)du \qquad \text{i. e.}$$

(3) $$\int_{\alpha}^{\beta} u\,dv = [B\beta - A\alpha] - \int_{A}^{B} v\,du.$$

Obviously (3) and (1) are equivalent.

We indicate now the extension to the slightly more general case where u is not a monotonic function of v and u is not necessarily positive on its entire range.

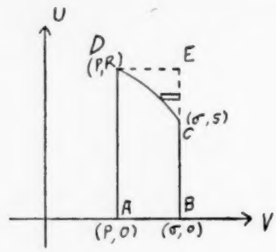
For simplicity we assume that du/dv exists (cf. footnote below) and is continuous over the closed interval $\alpha \leq v \leq \beta$. Split the curve over (α,β) into its greatest monotonic non-negative and non-positive parts indicated diagramatically.



It suffices to establish formula (3) for the three untreated cases:*

(i) u positive and monotonically decreasing.
(ii) u negative and monotonically increasing.
(iii) u negative and monotonically decreasing.

Case (*i*)



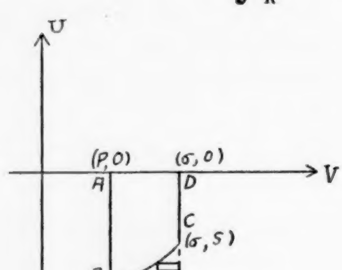
Clearly area $ABCD$ = area $ABED$ − area CED; i. e.,

$$\int_{\rho}^{\sigma} u\,dv = R(\sigma-\rho) - \int_{S}^{R} (\sigma - v)du$$

$$= R\sigma - R\rho - R\sigma + S\sigma + \int_{S}^{R} v\,du$$

$$= [S\sigma - R\rho] - \int_{R}^{S} v\,du.$$

Case (*ii*).



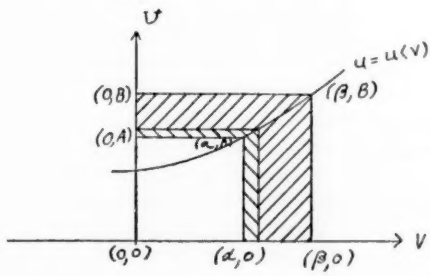
Here (the negative area $ABCD$) = (the negative area $ABED$) − (the negative area BEC); i. e.

$$\int_{\rho}^{\sigma} u\,dv = R(\sigma-\rho) - \int_{S}^{R} (\sigma - v)du$$

*We leave the trivial case in which u or v is a constant to the reader.

$$= [S\sigma - R\rho] - \int_R^S v\,du.$$

The treatment for case (iii) is now obvious.



In the course of a conversation after the above was written, Colonel Harris Jones pointed out that in the case first treated the following statements hold relative to the accompanying figure.

Shaded area $= B\beta - A\alpha$; consequently

$$\int_{x=a}^{x=b} u\,dv + \int_{x=a}^{x=b} v\,du = uv\Big]_{x=a}^{x=b}.$$

This demonstration is simpler geometrically than the preceding one. Furthermore, the corresponding differential relation

$$u\,dv + v\,du = d(uv)$$

is made geometrically plausible by a glance at the indicated differential strips, and the integral relation is rendered equally plausible by finite summation.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 515. Proposed by *N. A. Court*, University of Oklahoma.

Through a point M in the face BCD of the tetrahedron $ABCD$ planes are drawn respectively parallel to the faces ACD, ADB, ABC meeting the edges AB, AC, AD in the points P, Q, R. The lines MP, MQ, MR are produced to the points P', Q', R' so that $MP=2PP'$, $MQ=2QQ'$, $MR=2RR'$. Show that the plane $P'Q'R'$ passes through the vertex A.

The corresponding problem in the plane was considered in the *Journal de Mathématiques Élémentaires*, 1888, pp. 238-239, Q. 256.

Solution by *Paul D. Thomas*, U. S. Navy.

The planes PQR and $P'Q'R'$ are parallel. Let the plane PQR meet AM in M'. Now if A lies in $P'Q'R'$ it must be shown that $MM'=2AM'$.

The line of intersection of the two planes through M parallel respectively to ABD and ACD meets ABC in T. Similarly there is a point S in ACD, and U in ABD, so that the points A, P, T, Q, R, U, M, S are vertices of a parallelepiped circumscribing the tetrahedron $MPQR$ and having AM for diagonal. Thus the plane PQR meets AM in M' such that $MM'=2AM'$, i. e. MM' is a median of $MPQR$ and is equal to two thirds of AM. (See N. A. Court, *Modern Pure Solid Geometry*, p. 59.) Thus A lies in the plane $P'Q'R'$.

Also solved by *Frank Hawthorne*.

No. 518. Proposed by *H. Jones*, U. S. Military Academy.

Find the envelope of a family of circles which touch one of two perpendicular lines and intercept a segment of given length on the other.other.

I. Solution by the *Proposer*.

If the given lines are taken for coordinate axes, a circle (P) having for center the point $P(u,v)$, touching the Y-axis and intercepting a segment of given length $2c$ on the X-axis has for its equation.

$$(1) \qquad (x-u)^2+(y-v)^2=u^2,$$

where u and v satisfy the additional relation

$$(2) \qquad u^2-v^2=c^2.$$

(Thus the locus of P is an equilateral hyperbola having the points $(c,0)$ and $(-c,0)$ for vertices.)

Eliminating v from (1) and (2), we have

$$(3) \qquad (x-u)^2+[y-(u^2-c^2)^{\frac{1}{2}}]^2=u^2.$$

and differentiating (3) with respect to u, we obtain

$$(4) \qquad (x-u)(u^2-c^2)^{\frac{1}{2}}=-uy.$$

The result of eliminating x from (3) and (4) may be put in the form

$$(5) \qquad [y-(u^2-c^2)^{\frac{1}{2}}][y+c^2(u^2-c^2)^{\frac{1}{2}}/(2u^2-c^2]=0.$$

Then from (4) and (5) we obtain the two sets of equations

$$(6) \qquad x=0, \quad y=(u^2-c^2)^{\frac{1}{2}}$$

$$(7) \qquad x=2u^3/(2u^2-c^2), \quad y=-c^2(u^2-c^2)^{\frac{1}{2}}/(2u^2-c^2).$$

The equations (6) show that the Y-axis is a part of the envelope. which was obvious a priori, and the equations (7) give a parametric representation of the required envelope.

II. Solution by *Nev. R. Mind*.

Let (P) be a variable circle having its center on a given curve (H) and tangent at Q to a given curve (Y). To find the envelope (E) of (P) consider a second circle (P') having its center P' on (H), tangent to (Y) at Q', and cutting (P) in the points U, V. If (P') varies, approaching (P), the point Q' will approach Q, for (Y) is obviously a part of (E), and one of the points U, V, say U, will also approach Q. To find the limiting position of the point V, observe that the chord

UV of (P) is perpendicular to the line of centers PP' of the two circles, and that the limiting position of PP' is the tangent p to (H) at P. Thus the point common to (P) and (E) is the other end, R, of the chord of (P) passing through Q and perpendicular to the diameter p of (P).

Thus the envelope (E) may be constructed by points as follows. If the normal to (Y) at a point Q meets (H) in P, the reflection R of Q in the tangent to (H) at P lies on (E).

Simple cases of the above proposition are obtained when (H) and (Y) are two straight lines, or two concentric circles, etc.

In the case of the proposed problem, (H) is an equilateral hyperbola, and (Y) is the conjugate axis, y, of (H). The point Q is the projection of P upon y. Thus the required locus consists of the reflections in the tangents of (H) of the projections of the respective points of contact upon the conjugate axis of (H).

EDITORIAL NOTE. It may be of interest to observe that equation (4) of solution I derived by differentiation represents a straight line passing through the point $(0,v)$ and perpendicular to the tangent to (H) at $P(u,v)$. This line is therefore identical with the line QR obtained in Solution II as the limiting position of the common chord UV.—N. A. C.

No. 519. Proposed by *K. J. Nielsen*, Louisiana State University.

How many different arrangements can be made of two identical sets of n distinct cards if they are placed equally spaced around a ring?

Solution by *H. S. Grant*, Rutgers University.

We treat the following generalization: Find the number of cyclic permutations of $\sum_{i=1}^{n} x_i$ letters of which x_1 are alike, say A_1, x_2 others are alike, say $A_2, \cdots$, x_n others are alike, say A_n.

Solution: Let $(x_1, \cdots, x_n) = d$ denote the greatest common divisor of the x_i's. Let d' be any divisor of d and put $d_i = x_i/d'$. We consider the number of cyclic permutations of $\sum_{i=1}^{n} d_i$ letters consisting of d_i A_i's and obtained from those linear permutations of these letters which give rise to exactly $\sum_{i=1}^{n} d_i$ like cyclic permutations. This number is

$$P_{d'} = \Bigg[\frac{(\sum_i d_i)!}{d_1!\, d_2! \cdots d_n!} - \sum_{j=1}^{m} \frac{(\sum_i d_i/p_j)!}{(d_1/p_j)!(d_2/p_j)! \cdots (d_n/p_j)!}$$

$$+ \sum_{j,k} \frac{(\sum_i d_i/p_j p_k)!}{\Pi_i (d_i/p_j p_k p_l)!} - \sum_{j,k,l} \frac{(\sum_i d_i/p_j p_k p_l)!}{\Pi_i (d_i/p_j p_k p_l)!} + \cdots$$

$$\left.\pm \frac{(\sum_i d_i/p_1p_2\cdots p_n)!}{\prod_i (d_i/p_1p_2\cdots p_n)!}\right] \div \sum_i d_i,$$

where $(d_1, d_2, \cdots d_n) = d'' = p_1^{y_1}p_2^{y_2}\cdots p_m^{y_m}$, $y_s > 0$, p_s the distinct prime factors of d''. If $d'' = 1$, only the first term appears on the right.

Proof. For each linear arrangement of r letters, there are r cyclic arrangements which are alike, unless the arrangement consists of x identical groups of y letters each and $xy = r$. In the numerator of $P_{d'}$ we have given the number of arrangements of the $\sum_i d_i$ letters A_i which fall into groups of cyclic arrangements of which $\sum_i d_i$ are alike. For we have subtracted out just once all sub-groups of such arrangements which have a fewer number of like cyclic arrangements. This follows from the identity, $\sum_{i=1}^{z}(-1)^i {}_zC_i = -1$, where ${}_zC_i$ denotes the number of combinations of z distinct letters taken i at a time.

Considering all possible arrangements of the $\sum_i x_i$ letters into d' identical groups each consisting of d_i A_i's, the solution to our problem is obviously

$$P = \sum_{d'/d} P_{d'}.^*$$

Special Cases: If $x_1 = x_2 = \cdots = x_n = s$, then we have

$$P = \sum_{d/s}\left[\frac{(dn)!}{(d!)^n} - \sum_{j=1}^{m}\frac{(nd/p_j)!}{[(d/p_j)!]^n} + \sum_{j,k}\frac{(nd/p_jp_k)!}{[(d/p_jp_k)!]^n} - \cdots \right.$$

$$\left. \pm \frac{(nd/p_1p_2\cdots p_m)!}{[(d/p_1p_2\cdots p_m)!]^m}\right] \div dn,$$

where $d = p_1^{y_1}p_2^{y_2}\cdots p_m^{y_m}$, $y_s > 0$, p_s the prime factors of d. If $d = 1$, only the first term appears on the right. In particular, if $s = 2$, we have

$$P = \frac{n!}{(1)^n n} + \left[\frac{(2n)!}{2^n} - \frac{n!}{1^n}\right] \div 2n = \frac{(2n-1)!}{2^n} + \frac{(n-1)!}{2}.$$

No. 523. Proposed by *Paul D. Thomas*, U. S. Coast and Geodetic Survey.

Find the power of the incenter of a triangle with respect to the circumcenter, in terms of the sides of the triangle.

*By d'/d we mean "d', a divisor of d." Thus the summation is to be taken over all the divisors of d. Analogous interpretation for d/s in the second line following.

Solution by *Earl V. Greer*, Bethany-Peniel College, Bethany, Oklahoma.

The distance d between the circumcenter and incenter of a triangle is given by the relation $d^2 = R(R-2r)$, where R, r are the circumradius and inradius, respectively. Therefore

$$d^2 - R^2 = -2Rr = -(2abc/4S)(2S/p) = -abc/(a+b+c),$$

where S, p are respectively the area and the perimeter of the triangle. (Reference: N. A. Court, *College Geometry*, paragraphs 202, 200, 107.)

Also solved by the *Proposer*.

No. 524. Proposed by *J. Frank Arena*, Hardin, Illinois.

The equation $ax+by=c$ is to have exactly n distinct pairs of positive integral solutions (x,y). For given a,b, show that the greatest value of c is $(n+1)ab-a-b$, and that the least value of c is

$$(n-1)ab+a+b.$$

Solution by *H. T. R. Aude*, Colgate University.

That there is an inconsistency in the results submitted with the proposal is readily seen by considering a particular case, say $3x+4y=c$, $n=2$. By use of the stated results, the greatest and least values of c turn out to be 29 and 19. But consider the two equations $3x+4y=30$ and $3x+4y=18$. The first has the solutions (2,6) and (6,3). If however it is claimed that (10,0) is a third solution, implying that zero is to be included with the positive integers, then the second equation has two solutions (2,3) and (6,0). Thus, whether zero is to be included or not, the given values for c do not hold. Further, if a or b is 1 the proposed least value exceeds the proposed greatest value.

Discarding solutions which contain $x=0$ or $y=0$ and assuming, as usual, that a and b represent relatively prime, positive integers, consider the graph of the equation $ax+by=c$ when referred to rectangular coordinates. There exists a one-to-one correspondence between the positive integral solutions of the equation and those particular points $P_i(i=1,2,\cdots,n)$ of its graph which have positive integral coordinates. It is seen that if the point P_i with the integral coordinates (x_i,y_i) is on the graph, then, reading from left to right, the points $P_{i-1}(x_i-b,\ y_i+a)$ and $P_{i+1}(x_i+b,\ y_i-a)$ are respectively the points on the graph which come immediately before and after P_i. If n solutions exist, then the distance in the x-direction from the first solution represented by the point $P_1(x_1,y_1)$ to the nth solution represented by $P_n(x_n,y_n)$ is

$$x_n - x_1 = (n-1)b,$$

while, similarly $$y_1 - y_n = (n-1)a.$$

The value of c will be least when P_1 and P_n are located, respectively, as close as possible to the y-axis and x-axis. That is when $x_1 = 1$ and $y_n = 1$. This leads to the two relations

$$x_n - 1 = (n-1)b, \qquad y_1 - 1 = (n-1)a.$$

The least value of c for exactly n solutions, which may be symbolized by c_n(least), is found by substituting the coordinates of P_1 (or of P_n) just given, in the equation $ax+by=c$. It turns out to be

$$c_n(\text{least}) = (n-1)ab + a + b. \tag{1}$$

On the other hand the value of c will be greatest when P_1 and P_n respectively, have the coordinates (b, y_1) and (x_n, a). This situation leads to the two relations

$$x_n - b = (n-1)b, \qquad y_1 - a = (n-1)a$$

and thus to

$$c_n(\text{greatest}) = (n+1)ab. \tag{2}$$

The relations in (1) and (2) answer the questions of the problem. But, while it is true that every value of c_n (i. e., a value of c which belongs to an equation with exactly n solutions) is found on the range defined by (1) and (2), namely,

$$(n-1)ab + a + b \leqq c \leqq (n+1)ab, \tag{3}$$

it is not true for all cases that every c on the range given by (3) is of the class (c_n), where the class thus designated will include all the values of c_n. To make this clear, consider first the expressions

$$c_{n-1}(\text{greatest}) = nab, \qquad c_{n+1}(\text{least}) = nab + a + b, \tag{4}$$

obtained from (2) and (1), and arrange in order the expressions given in (1), (2) and (4).

It is not difficult to show that (a) $a+b$ exceeds ab if and only if $a=1$ or $b=1$, whence ab and $a+b$ are consecutive integers; (b) $a+b=ab$ is possible only if $a=b=2$, which case is excluded since a and b are relatively prime; (c) ab exceeds $a+b$ for all other values of a and b. Thus if $a=1$ or $b=1$, then the order is

$$nab < (n-1)ab + a + b < (n+1)ab < nab + a + b,$$

or $$c_{n-1}(\text{greatest}) < c_n(\text{least}) < c_n(\text{greatest}) < c_{n+1}(\text{least.})$$

Hence every c on the range given by (3) is of the class (c_n). For all other values of a, b the order is

$$(n-1)ab + a + b < nab < nab + a + b < (n+1)ab$$

or $$c_n(\text{least}) < c_{n-1}(\text{greatest}) < c_{n+1}(\text{least}) < c_n(\text{greatest}).$$

The elements c on the range defined by (3) are seen to belong to the three classes (c_{n-1}), (c_n), (c_{n+1}).

To sum up, when $a=1$ or $b=1$ the various classes are separated and follow each other without any gaps, but in all other cases for any one class (c_n) there is infiltration at the respective ends with elements from the classes (c_{n-1}) and (c_{n+1}). It is true that in the middle of the range in (3) there exists a limited range of $a+b-1$ elements defined by

$$nab+1 \leqq c \leqq nab+a+b-1$$

where every c is of the class (c_n).

The problem, with the proposed erroneous values of c, is taken from Hall and Knight, *Higher Algebra*, (4th Edition), p. 291.

No. 529. Proposed by *William Taylor*, Colgate University.

Consider a field piece firing shells at a constant muzzle velocity but with varying angle of elevation. Prove that the vertices of the paths of these shells form an ellipse. (Neglect air resistance.)

I. Solution by *A. Sisk*, Maryville, Tennessee.

Eliminate t from the usual equations of motion,

$$y = -\tfrac{1}{2}gt^2 + v_0 t \sin \Theta, \qquad x = vt \cos \Theta,$$

where Θ is the variable angle of elevation and v_0 is the constant muzzle velocity, and get

$$y = (\tan \Theta)x - (g/2v_0^2 \cos^2\Theta)x^2. \tag{1}$$

Then, at the vertex of the parabolic path

$$dy/dx = \tan \Theta - gx/v_0^2 \cos^2\Theta = 0, \tag{2}$$ whence

$$x = v_0^2 \sin 2\Theta/2g. \tag{3}$$

With this value for x, (1) becomes, after easy reductions,

$$y = v_0^2 \sin^2\Theta/2g. \tag{4}$$

Equations (3) and (4) constitute parametric equations of the desired ellipse. Elimination of the parameter gives the rectangular equation,

$$x^2 + 4y^2 - 2v_0^2 y/g = 0.$$

II. Solution by *Frank Hawthorne*, Allegheny College, Meadville, Pennsylvania.

The common directrix of the parabolic paths is horizontal at altitude $v_0^2/2g$. Hence the locus of the foci of the paths is a circle with

radius $v_0^2/2g$ and center at the gun. (The locus of the foci being a circle and the existence of a common directrix are considered in some detail in Synge and Griffith, *Principles of Mechanics*, pp. 150-152.) From this, the locus of the vertices must be the figure composed of all the midpoints of lines drawn from the circle perpendicular to the directrix. Thus we have a locus in which the ordinates (measured from the directrix) are each half the corresponding ordinates of the circle. It is thus an ellipse with center at $(0, v_0^2/4g)$, and with semi-axes equal to $v_0^2/2g$ and $v_0^2/4g$, the semi-major axis being horizontal.

J. F. Kenney found the problem in March and Wolff, *Calculus* (3rd ed.) p. 88. Also solved by *Ferrel Atkins, Leon Shenfil, Paul D. Thomas* and the *Proposer*.

PROPOSALS

No. 554. Proposed by *N. A. Court*, University of Oklahoma.

The two lines which join the circumcenter of a triangle and its isotomic to the Lemoine point and the orthocenter, respectively, are parallel.

No. 555. Proposed by *Walter B. Clarke*, San Jose, California.

A transversal meets the sides BC, CA, AB of a triangle ABC in the points P, Q, R; the lines AP, BQ, CR meet the circumcircle (O) of ABC in the points P', Q', R'; the lines $A'P'$, $B'Q'$, $C'R'$ joining P', Q', R' to the diametric opposites of A, B, C, on (O), meet BC, CA, AB in the points A_0, B_0, C_0. Show that the lines AA_0, BB_0, CC_0 are concurrent.

No. 556. Proposed by *Howard D. Grossman*, New York City.

Prove that the limit, as $n\to\infty$, of the ratio

$$(n+1)(p+q)^{C}(n+1)p : n(p+q)^{C}np$$

is $(p+q)^{p+q}/p^pq^q$.

No. 557. Proposed by *E. P. Starke*, Rutgers University.

A boy makes two snowballs, one having twice the diameter of the other. He brings them into a warm room and lets them melt. When the larger one is half melted, how much is left of the smaller?

No. 558. Proposed by *Frank C. Gentry*, University of New Mexico.

A variable triangle has a fixed base and the difference of its base angles is constant. Show, both analytically and synthetically, that the locus of the third vertex and the locus of the orthocenter are identical.

No. 559. Proposed by *N. A. Court*, University of Oklahoma.

At each vertex of a tetrahedron planes are drawn perpendicular to the edges passing through that vertex. The twelve planes thus obtained may be grouped to form four parallelepipeds. Show that their sixteen diagonals have a point in common.

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Basic Mathematics. By Walter W. Hart. Boston, D. C. Heath and Company, 1942. vi+456 pages. $1.52.

The author, who is one of known great competence, provides an excellent survey (2 parts=4+5 chapters=14+16 main lettered subdivisions=132+398 sections =176+262 pages) of elementary and secondary mathematics, which is widely used (about 65,000 copies to date) in the 11th and 12th grades and also in some war courses. The first six chapters (referred to hereafter as Part I+), (210 pages), are published separately; these cover arithmetic, elementary geometry and algebra, logarithms, and the trigonometry of right triangles, all approached, for the most part, by rule and intuition. The remaining three chapters (Part II−), (228 pages), deal with demonstrative plane and solid geometry, and advanced algebra. The range of material covered in a given topic is sometimes considerable, as, for example, from reference to a plane only as a flat surface in the geometry of Part I+ to the use of Cavalieri's Theorem in deriving the volume of a sphere in that of Part II−, and from algebra which does not even include the addition of the most simple algebraic fractions in Part I+ to simultaneous quadratics, progressions and the binomial theorem in Part II−. The instruction contained in Part I+, which may be terminal at an elementary level, was "suggested by the author's investigations" (including visiting various trade schools) "of current needs supported by interpretations of these needs as found in Bulletin TM 1-900 of the War Department and Leaflet #62 of the Office of Education." Learning is stimulated, particularly in this part, by pictures and silhouettes of technically designated popular war planes as well as by a variety of scale drawings; roughly 150 and 25 of these types, out of totals of 250 and 350 figures in Part I+ and Part II−, respectively, suggest the great motivation present especially in Part I+. Problems and exercieses whose parts total about 3000 and 2000 in Parts I+ and II−, respectively, often furnish interesting incidental information. Discussion and problems about wing cambers, coefficients of lift and drag, wind-drift, composite graphs, time zones, charts, map projectiohs, etc., make the author's "special indebtedness to Bulletins 24 and 26 of the Civil Aeronautics Administration" understandable.

Meticulous attention to technical detail is in evidence. The handbook size $4^5/_8$x$6^7/_8$ inch pages, all full to their margins, are bound between heavy cloth-covered boards by sturdy oversewing on tapes. Chapters and their main lettered sub-divisions, which are referred to by pages in the table of contents, begin invariably at the tops of pages, but the numbered sections, indicating only slight changes of thought, sometimes break into sets of exercises or problems. There are approximately 325 such sets each entirely on one page, with the exception of the two longer review sets. Italics is used for emphasis and in statements of directions, rules and theorems. Boldface type is used for new terms at their points of definition; all of these are included among over 400 items indexed on 5 pages. Tables include 6-place decimal equivalents of 64ths, squares and square roots as well as cubes and cube roots of all integers from 1 to 200, and 4-place logarithms of numbers as well as values and logarithms of trigonometric functions.

Specific references to the text are made in the statements of minor importance which follow. The procedure for rounding off products of approximate numbers (p. 10)

is somewhat obscure; it gives, for example, answers of 7.7 and 394.4 in exercises 19 and 23 respectively, even though two 3-place numbers are multiplied in each case. Again (p. 182, l. 17), that 3 places "is the limit of accuracy of a 4-place table" is a pessimistic view. The inclusion of involution and evolution among the fundamental operations of algebra (p. 108) is unusual. The rule involving equivalent equations resulting from evolution (p. 132) is not entirely clear. "Axioms" 9 to 13 (p. 243) are consequences of others given previously and the definitions of the symbols $>$ and $<$. Some teachers might question the postulates stated on pages 347 and 350. The "common solutions" of a system (pp. 409 and 413) are preferably called "solutions". It is customary to omit the commas around symbols such as T in "The total area, T, of a cube" (p. 97).

Among parts of scale figures which appear to be slightly inaccurate are: the angle 60° (1st figure, p. 57); the angle 70° (1st, p. 79); the vertical 10″ (7th, p. 90); and the 30″ between centers (1st, p. 137). Misprints occur rarely but there are some. After 375″ (p. vi), add "$= {}^3/_8$"; the dimension ${}^{13}/_{24}$″ (p. 24) is incorrectly indicated; "Example" should read "Illustration" (p. 24, l. 31); "positive" should read "negative" (p. 111, l. 2); "50" should read "50°" (p. 201, l. 8); the digit in the 4th decimal place in the number should be 5 (p. 290, l. 2); the words "(or more)" should be omitted (p. 381, l. 1). The original answer book of 53 pages has been corrected and will require further revision.

The Municipal University of Omaha. JAMES M. EARL.

Mathematics for Navigators. By Delwyn Hyatt and Bennett M. Dodson. McGraw-Hill Book Company, New York, 1944. vii+106 pages. $1.25.

Commanders Hyatt and Dodson, being of and for the Navy, have done an excellent bit of nautical work in their introduction-review, *Mathematics for Navigators.* True to Naval custom, these men have employed the time-proven system used by sea-going men; they reflect the attitude of "those who do," as contrasted, favorably, with those who theorize only.

This is a good, short, practical summary of those mathematical functions which must be understood thoroughly by all who would make a study of the theory of navigation, if they propose to become at all expert in this science. Although your reviewer has navigated over a million miles of surface and aerial travel, and has taught the basic theory of navigation for several years, he finds that his careful scanning of *Mathematics for Navogators* is a decidedly beneficial review of those fundamentals so necessary to the correct understanding of navigational principles.

It seems to this reviewer that the authors have combined a psychological presentation of instruction with a logical presentation of factual data. These data are neither "cut-and-dried" nor simplified; they are, in fact, data which are readily used in regular solutions of actual and applied navigational problems. The transition from Arithmetic through Algebra, Logarithms, Plane Geometry, Plane Trigonometry, and Spherical Trigonometry to Oblique Spherical Triangles is logically sound, and it can be made readily by anyone who has a basic knowledge of simple mathematics.

Although neither Analytical Geometry nor Descriptive Geometry is mentioned in this text, it is firmly felt that an introductory course in each would be useful to a student of navigation, inasmuch as such courses do aid the student in visualizing three-dimensional objects in their proper space relation. Especially does this appear indicated as a preparation for the study of ***Oblique Spherical Triangles*** and Functions of the Celestial Sphere.

A prospective student of navigation should not infer from this review that the text in question is a complete treatise on mathematics or navigation. On the contrary, the authors have not tried to write another exhaustive text on either of the above sciences; they have presented their work "for the assistance of those who may by studying navigation 'on their own,' without sufficient mathematical background." One may add that "It should also prove useful to those who merely need to review their mathematics in order to pursue the study of navigation with confidence."

Navigation, as such, cannot be taught unless actual navigation is experienced and practiced. Preparation for the study of navigation must include an applicable knowledge of simple mathematics. If anyone desires to study the theory of navigation—later, perhaps, to apply the studied theory to the actual practice of navigation—he would do well to review his mathematics carefully. If he does not have a mathematical background to review, he may become acquainted with the basic and necessary fundamentals of mathematics, in so far as the theory of navigation is concerned, by a careful study of *Mathematics for Navigators.*

Errata: Page 3, fourth line from the bottom should read "... required to go 13 miles," not "... 13.5 miles..."
Page 49, top of page, a "t" has been omitted in the spelling of "hypotenuse."

Evanston, Illinois. E. L. EDMONDSON.

INSTITUTE FOR MATHEMATICS TEACHERS

GRADES 7-12

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"Visual Aids in the Study of Mathematics" (six lectures). Leader: Professor E. H. C. Hildebrandt, Professor of Mathematics Northwestern University.

Professor Hildebrandt is recognized throughout the nation for his work in this field.

UNIT II JULY 6, 7, 8

"Air and Sea Navigation" (five lectures). Leader: Professor F. G. Dressel, Mathematics Department, Duke University.

Professor Dressel has been conducting courses in Navigation in the Navy Program at Duke University.

UNIT III JULY 4-14

"Mathematical Education" (nine lectures without conflict with other units). (Emphasis on war and post-war planning). Leader: Miss Veryl Schult, Director of Mathematics in the High Schools of Washington, D. C. Miss Schult is well known in the councils of mathematics teachers.

Professor Hildebrandt, Schlauch, Scates, and others will assist Miss Schult in conducting this unit of work.

UNIT IV JULY 10, 11. 12

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Mr. A. L. M. Wiggins, President of the American Bankers Association will address the Institute on July 12, 8:15 P. M.

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